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ABSTRACT

Inertial range spectra of turbulent flows are deduced from the equations of motion. An explanation for the validity of universal equilibrium theories is provided. It is shown that Fourier-space cascades and physical-space intermittence are complementary aspects of the same reality. Spectra of the form $E(k) = Ak^{-\sigma}$ are found, where k is the wave number, A is independent of k, and σ is a positive constant. σ is evaluated; for the Burgers equation $\sigma = 2$; for the Navier-Stokes equations in both two and three space dimensions $\sigma = 2.3 \pm 0.15$.

Introduction

The velocity field of a fluid in turbulent flow is the sum of many components of widely differing scales. A fundamental objective of the theory of turbulence is to analyze the effect of the smaller scales on the larger ones, so that this effect can be represented by an appropriate eddy coefficient in a closed system of equations capable of solution by standard means. The first step towards this goal is the study of the smaller scales of motion by themselves; in many cases of physical interest the important scales of motion belong to the inertial range, which we shall now define.

Let $\underline{v}$ be the velocity vector in a turbulent flow, with components $v_i = v_i(\underline{x})$; x_1, x_2, x_3, components of the vector $\underline{x}$, are the space variables. Vector quantities will be underlined, and averages (whose exact meaning remains to be defined), will be denoted by bars above the averaged quantities.

We write

$$v_i(\underline{x}) = \overline{v}_i(\underline{x}) + u_i(\underline{x}) .$$

The correlation tensor $R_{ij}(\underline{x},\underline{r})$ is defined by (see [1])

$$R_{ij}(\underline{x},\underline{r}) = \overline{u_i(\underline{x})u_j(\underline{x}+\underline{r})} ;$$

R_{ij} has a Fourier transform Φ_{ij} given by

$$R_{ij}(\underline{r}) = \int \Phi_{ij}(\underline{k})e^{i\underline{k}\cdot\underline{r}}\, d\underline{k}$$

where the dependence of R_{ij} and Φ_{ij} on $\underline{x}$ is not explicitly mentioned. Let k be the length of the vector $\underline{k}$, and let $dA(\underline{k})$ be the surface element on a sphere $S(k)$ of radius k centered at the origin. Write

$$\Psi_{ij} = \int_{S(k)} \Phi_{ij}(\underline{k})dA(\underline{k}) .$$

The energy spectrum E(k) is defined by

$$E(k) = \frac{1}{2} \Psi_{ii}(k) \qquad \text{(the summation convention is in use)}$$

and one has the obvious relations

$$\text{Average energy at } \underline{x} = \frac{1}{2} \overline{u_i(\underline{x})u_i(\underline{x})} = \frac{1}{2} R_{ii}(0)$$

$$= \int_0^\infty E(k)dk .$$

It can be shown that E(k) is non-negative.

We assume the flow is described by the Navier-Stokes equations

$$\partial_t \underline{v} + (\underline{v}\cdot\underline{\nabla})\underline{v} + \text{grad } p = \nu\Delta\underline{v}$$

$$\text{div } \underline{v} = 0$$

where t is the time, p is the pressure and ν is the viscosity. From these equations it can be seen that ε, the rate of energy dissipation, is

$$\varepsilon = 2\nu \int_0^\infty k^2 E(k)dk$$

$2k^2E(k)$ is the spectral dissipation function.

In flows at high Reynolds number $E(k)$ differs from zero over a wide range of k's, and furthermore the energy spectrum $E(k)$ and the spectral dissipation function $2k^2E(k)$ are approximately disjoint, i.e. there exist numbers k_1, k_2 (not uniquely defined), such that

$$\int_0^{k_1} E(k)dk \cong \int_0^{\infty} E(k)dk \ ,$$

$$\int_{k_2}^{\infty} k^2E(k)dk \cong \int_0^{\infty} k^2E(k)dk$$

and

$$k_1 << k_2 \ .$$

The intervals $(0,k_1)$, (k_1,k_2), and (k_2,∞) are respectively the energy (or energy-containing) range, the inertial range, and the dissipation range.

Attempting to find general expressions for $E(\underline{x},k)$, k in the energy range, is both impossible and useless, for the following reasons: the dependence of E on $\underline{x}$ leads to problems with a prohibitive number of independent variables, and the dependence of E on the initial and boundary data makes the existence of interesting general results unlikely. The study of $E(k)$, k in the energy range, should be carried out as outlined in the opening paragraph: one should replace the effects of the inertial and dissipation ranges by appropriate eddy coefficients and solve the resulting energy range equations in each special case. Steps in this direction have been described, i.a. in [10] and [13].

The study of the inertial and dissipation ranges is a sounder undertaking. Roughly speaking, energy reaches these ranges after a number of non-linear interactions; the influence of the boundary and initial conditions is no longer appreciable; over most of the flow the spectrum becomes independent of $\underline{x}$ and can be expected to embody an intrinsic property of turbulence. It is of course a task of a theory of turbulence to provide a more credible explanation of this lack of dependence between the several parts of the spectrum. When the Reynolds number is sufficiently large the inertial range is of considerable extent, and its influence on possible eddy coefficients is dominant. (See e.g. [11] and a forthcoming paper by the present author.) The present paper is devoted to the study of the inertial range.

The Kolmogoroff Law

The following widely quoted result in the theory of the inertial range is due to Kolmogoroff. Let ε be the rate of energy dissipation,

$$\varepsilon = 2\nu \int_0^\infty k^2 E(k)\,dk \ .$$

If it is assumed that $E(k)$, k in the inertial range, may depend only on ε and k (and not, for example, on the viscosity or the initial data) then, on dimensional grounds,

$$E(k) = \text{constant}\cdot\varepsilon^{2/3}\, k^{-5/3} \ .$$

Theories of turbulence tend to be judged by their ability to reproduce this law in three-dimensional flow; yet its validity would be more a puzzle than an explanation. The dimensional argument is independent of the equations of motion and should be valid for other equations; we shall see in the next section that this is not the case. The validity of Kolmogoroff's law implies that there is no statistical dependence between large and small scales of motion and that at high Reynolds numbers the rate of energy dissipation is independent of viscosity (see the excellent discussion in [12]). It is far from obvious that this should be true. The hypotheses on which Kolmogoroff's law is based are in no way necessary for the existence of an inertial range. It is therefore of interest to evaluate the inertial range spectrum from a dynamic theory, i.e. using the Navier-Stokes equations or a credible theory of turbulence.

Kraichnan [7] has performed such a computation on the basis of his direct interaction approximation and obtained for three-dimensional flow

$$E(k) = \text{constant}\cdot k^{-3/2} .$$

Kraichnan's later work is not relevant in this context, since it is based on the assumption that Kolmogoroff's law is valid and does not afford an independent check of the latter. The author knows of no other previous attempt to verify Kolmogoroff's law.

Several inertial range spectra for two-dimensional turbulence have been proposed of late, also on the basis of dimensional arguments. The energy spectrum

$$E(k) = \text{constant}\cdot k^{-3}$$

has been derived by Leith and by Kraichnan (see [8], [9]). It is interesting to note that all proposed laws are of the form

$$E(k) = \text{constant}\cdot k^{-\beta} \quad , \quad \beta \text{ constant} .$$

A discussion of these expressions will be presented later.

The method described in this paper can be summarized as follows: The Fourier transform $\hat{\underline{u}}(\underline{k},t)$ of $\underline{u}(\underline{x},t)$ is taken. (An outline of a justification of this step can be found in [1].) An equation of motion for $\hat{u}(\underline{k},t)$ is derived from the Navier-Stokes equations and the assumption that $|\underline{k}|$ is the inertial range. The resulting equations are essentially transforms of the Euler equations, subject to certain restrictions on the possible form of the solution. Stationary solutions of these equations are then sought. The reason for singling out stationary solutions is best brought out by an analogy with a finite mechanical system: Consider such a mechanical system with N degrees of freedom denoted by the variables $\alpha_1,\ldots,\alpha_N$, and whose motion is described by equations of the form

$$(1) \qquad \frac{d\alpha_i}{dt} = F_i(\alpha_1,\ldots,\alpha_N)$$

with initial conditions

$$\alpha_i(0) = g_i$$

where the g_i are random with an unknown distribution law. If a stable stationary solution α_i^o of (1) can be found, to which all solutions will tend, then at time t large enough the average values of α_i will be approximated by α_i^o. The appropriate average is obviously an average over an ensemble of realizations of the system.

For the sake of clarity, we shall first apply our method to the one-dimensional Burgers equation. This is the object of the following section.

The Burgers Equation and its Inertial Range

Consider the Burgers equation

$$(2) \qquad \partial_t u + \frac{1}{2}\,\partial_x(u^2) - \frac{1}{R}\,\partial_x^2 u = 0\ , \quad R \text{ constant}\ .$$

The behavior of the solution of (2) can be studied by fairly standard methods. (In fact (2) can be reduced to a linear equation and solved, but this will be of little use in the present context.) Solutions of the "inviscid" equation

$$\partial_t u + \frac{1}{2}\,\partial_x(u^2) = 0$$

are obtained, and rendered single-valued by apt insertion of shocks. The structure of these shocks can be determined and the energy

spectrum as well as the dissipation function approximately evaluated (see [12]). An inertial range is found, with

$$(3) \qquad E(k) = Ak^{-2}$$

where A is independent of k (but does depend on the structure of the initial data). The Kolmogoroff hypothesis is thus invalid.

We now proceed to rederive (3) by a method capable of extension to fluid flow in several space dimensions. Let $\hat{u}(k,t)$ be the spatial Fourier transform of $u(x,t)$. It satisfies

$$(4) \qquad \partial_t \hat{u}(k,t) = -\frac{ik}{2}\int \hat{u}(k',t)\hat{u}(k-k',t)dk' - \frac{1}{R}k^2\hat{u}(k,t) .$$

The real character of u is reflected in the condition

$$(5) \qquad \hat{u}(k,t) = \hat{u}^*(-k,t)$$

where the asterisk denotes a complex conjugate function. Let k belong to the inertial range. By hypothesis, the dynamic effect of the viscosity R^{-1} on $\hat{u}(k,t)$ is negligible compared to the inertia terms. The last term in (4) can be neglected, yielding

$$(6) \qquad \partial_t \hat{u} = -\frac{ik}{2}\int \hat{u}(k',t)\hat{u}(k-k',t)dk' .$$

It should be emphasized that the neglect of the dissipation term in (4) is essentially different from the Kolmogoroff assumption that $E(k)$ is independent of R^{-1}. The equations are strongly non-linear, and a strong dependence on R in other parts of the spectrum

may strongly affect the amplitude of E in the inertial range. We shall in fact find spectra different from Kolmogoroff's. The fact that energy is dissipated at large k leads one to expect that

$$\lim_{k \to \infty} |\hat{u}(k)| = 0 .$$

This expectation will be fulfilled.

We note that equation (6) is invariant under the transformation

$$(7) \qquad \begin{aligned} \hat{u}' &= \hat{u}/U \\ k' &= k/K \\ t' &= t(UK) . \end{aligned}$$

In particular, choose K to be a wave number in the inertial range, and $U = \hat{u}(K)$. Having assumed that the energy is contained mostly in the energy range, we expect that $\hat{u}/U$ will become large as k/K decreases; if (6) is considered to be an equation for $\hat{u}'$, this leads to the condition

$$(8) \qquad \lim_{k \to 0} |\hat{u}(k)| = \infty .$$

The integral in (6) is necessarily singular, and must be interpreted in the obvious generalized sense

$$\int \hat{u}(k')\hat{u}(k-k')dk' = \lim_{\varepsilon \to 0} \left[\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{k-\varepsilon} + \int_{k+\varepsilon}^{\infty} \right) \hat{u}(k')\hat{u}(k-k')dk' \right]$$

One can write (6) and (8) in the symbolic form

$$\partial_t \hat{u} = F[\hat{u}] \tag{9}$$

where $F[\hat{u}] = F[\hat{u}](k)$ is a functional of $\hat{u}$ and a function of k.

We now seek steady solutions of (9), i.e. solutions of

$$-ik \int \hat{u}(k',t)\hat{u}(k-k')dk' = 0 \ .$$

In the present one-dimensional case such a solution is readily found. Let $\tilde{u}(x,t)$ be the function of which this solution is a transform. Obviously, we have

$$\partial_x(\tilde{u}^2) = 0$$

and

$$\tilde{u}(x) = \pm C \ , \quad C \text{ constant} \ , \tag{10}$$

different signs being assumed on separate intervals of the x-axis. From this one concludes that

$$\hat{u}(k) = \text{constant}\cdot k^{-1} \tag{11}$$

and

$$E(k) = Ak^{-2} \ .$$

The solutions (10) and (11) are worth examining. $\tilde{u}$ represents an array of shocks with arbitrary locations, i.e. an intermittent flow. $\hat{u}$ describes a wave number space cascade; indeed, if $0 < k' < k$, then

$$\frac{1}{k'}\,\frac{1}{k-k'} > 0$$

and if $k' > k$ or $k' < 0$

$$\frac{1}{k'}\,\frac{1}{k-k'} < 0 \; .$$

Thus $\hat{u}(k,t)$ is fed by modes with wave numbers smaller than k and depleted by interactions with wave numbers larger than k. Intermittent flows and wave-number space cascades appear as complementary aspects of the same reality, not unlike waves and particles in quantum theory.

A Numerical Method

It will now be shown how (11) can be derived by a method applicable to multidimensional problems as well. We seek stationary solutions of (9); these can be determined only up to an arbitrary constant. If the energy spectrum E(k) is to be unique except for a multiplicative constant, we must have

$$E(k/K) = \hat{u}(k/K)\hat{u}^*(k/K) = \text{constant}\cdot\hat{u}(k)\hat{u}^*(k) = \text{constant}\cdot E(k) \; .$$

Put

$$\hat{u}(k) = a(k) \exp\,(i\phi(k)), \quad a(k) \text{ real and positive} \; .$$

It follows that

$$a(k/K) = \text{constant}.a(k)$$

i.e., in view of (8)

$$a(k) = \text{constant}\cdot|k|^{-\beta} \; , \quad \beta > 0 \; . \tag{12}$$

We are in effect looking for similarity solutions of (6).

Consider now the possible forms of $\phi(k)$. ϕ must be an odd function of k, to satisfy (5). For $F[\hat{u}]$ to be finite, $\phi(k') + \phi(k-k')$ must change by π at $k' = 0$ and $k' = k$; this will cause the singularities in $\hat{u}(k')\hat{u}(k-k')$ to cancel. Furthermore, $\phi(k)$ must be such that

$$(13) \qquad \phi(k') + \phi(k-k') = g(k) + h(k'/k)$$

where g depends only on k and h only on k'/k. The significance of this last condition will become apparent shortly. Functions ϕ which satisfy all these conditions are

$$(14) \qquad \phi(k) = ak - \mathrm{sgn}(k)\pi/2 , \qquad a \text{ an arbitrary constant, } \mathrm{sgn}(k) = k/|k| .$$

This represents a natural choice of ϕ in view of the following observation: Consider an initial value problem for equation (4) with initial values consisting of a single Fourier mode,

$$\hat{u}(k) = \delta(k-1) \exp (i\psi \,\mathrm{sgn}(k))$$

i.e.

$$\tilde{u} = \cos (x+\psi) .$$

It can be verified that for all t

$$\hat{u}(k,t) = \sum_n b(k,t)\delta(k-n) \exp\ (in\psi + (n-1)\ \mathrm{sgn}(n)\pi/2)$$

$$= \sum_n b(n,t)\delta(k-n) \exp\ (ia\psi - \mathrm{sgn}(n)\pi/2)\ ,$$

$$a = \psi + \pi/2\ .$$

This form of ϕ has already been discussed in [4].

Given ϕ, there remains only the task of determining β. If $F[\hat{u}](k_o) = 0$ for some k_o, then $F[\hat{u}](k) = 0$ for all k; this is ensured by the condition (13) and follows from the following identities:

$$F[\hat{u}](k) = \text{constant}\cdot\int |k'|^{-\beta}|k-k'|^{-\beta}$$

$$\cdot \exp\ (iak - i(\pi/2)(\mathrm{sgn}(k') + \mathrm{sgn}(k-k')))dk'$$

$$= \exp\ (iak) \lim_{\varepsilon\to 0}\left[-\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{k-\varepsilon} - \int_{k+\varepsilon}^{\infty}\right] (|k'|^{-\beta}|k-k'|^{-\beta})dk'\ .$$

Putting

$$k' = \frac{k}{k_o} k''\ ,$$

we obtain

$$F[\hat{u}](k) = \exp\ (iak)\ \frac{k}{k_o}\left(\frac{k_o}{k}\right)^{-2\beta} \lim_{\varepsilon\to 0}\left[-\int_{-\infty}^{-\varepsilon k_o/k} + \int_{\varepsilon k_o/k}^{k_o-\varepsilon k_o/k} - \int_{k_o+\varepsilon k_o/k}^{\infty}\right]$$

$$\cdot |k''|^{-\beta}|k_o-k''|^{-\beta}dk''$$

or

$$F[\hat{u}](k) = \exp\ (ia(k-k_o))\left(\frac{k_o}{k}\right)^{-2\beta-1} F[\hat{u}](k_o)\ .$$

The problem is therefore reduced to finding β such that

(15) $$F[\hat{u}^{\beta}](1) = 0$$

where

$$\hat{u}^{\beta} = k/|k|^{\beta+1} = e^{i\pi/2}|k|^{-\beta} \exp(-\mathrm{sgn}(k)\pi/2) ,$$

and a was set equal to zero without loss of generality. The point $k_o = 1$ is chosen so that all the $\hat{u}^{\beta}(k_o)$ have the same amplitude, and information about the location of the zero can be obtained from the variation of the residuals

$$r_{\beta} = i\partial_t \hat{u}^{\beta}(1) = iF[\hat{u}^{\beta}](1) .$$

In Table I the results of such a computation are exhibited. The integration was performed as follows: Let

$$f(k') = \hat{u}^{\beta}(k')\hat{u}^{\beta}(1-k') ;$$

the generalized integral

$$\int_{-\infty}^{+\infty} f(k')dk'$$

is replaced by the equivalent expression

$$\int_{0}^{1} g(k')dk' , \qquad g(k') = \sum_{j=-\infty}^{+\infty} f(j+k')$$

g can be shown to be a bounded function of k'. $g(k')$ is approximated by

Table I

<u>Minimization of $F[\hat{u}^{\beta}]$</u>

$M = 100;\ e = 10^{-10};\ k_o = 1$

β	$r_{\beta,N}$	$\sigma_{\beta,N}$
0.90	.9747	.096
0.95	.4794	.023
0.98	.1896	.0036
0.99	.0946	.000092
1.00	-.0044	.0000020
1.01	-.1027	.0010
1.02	-.1984	.0040
1.05	-.5056	.026
1.10	-.9955	.10

$$g_N(k') = \sum_{-N}^{+N} f(j+k') ,$$

where N is chosen so that

$$|f(N-1)| + |f(-N+1)| \leq e , \quad e \text{ small} .$$

g_N was then integrated by a Monte-Carlo procedure. In Table I $r_{\beta,N}$ is the result of this integration and $\sigma_{\beta,N}$ the corresponding variance. The results show that $\beta = 1$ is the appropriate value of β and the result of the preceding section is recovered. We have

$$E(k) = Ak^{-2} .$$

The question of the stability of these stationary solutions remains to be answered. The conclusion that they are stable can be drawn from the existence of known solutions of (4) which for large t and large k have the spectrum (3), see [2], [3] and [12]. The numerical evidence drawn from finite difference integrations of (4) also points to this conclusion. The phase function ϕ appears clearly; in fact this is how it was originally found. With good will and a large number of Fourier modes in the calculation, the spectrum (3) can also be fleetingly observed; it is soon destroyed by the combined effects of a finite number of Fourier modes and a finite time step.

The reason for the stability of these solutions is easy to visualize in x-space. An unsteady solution will consist of a number of shock waves of differing intensity and differing speeds.

Some will coalesce, some will disappear to infinity, until there remain only shocks of equal intensity and speed.

The Inertial Range in Turbulent Incompressible Flow

Let $\hat{\underline{u}}(\underline{k},t)$, with components $\hat{u}_\alpha(\underline{k},t)$, be the Fourier transform of the fluctuating velocity vector $\underline{u}(\underline{x},t)$; the Fourier transform of the Navier-Stokes equations is, in non-dimensional form,

$$(16) \qquad \partial_t \hat{u}_\alpha(\underline{k},t) = -ik_\delta P_{\alpha\gamma} Q_{\delta\gamma} - \frac{k^2}{R}\hat{u}_\alpha , \qquad \hat{u}_\alpha(-k) = u^*_\alpha(k)$$

where

$$Q_{\alpha\gamma} = \int \hat{u}_\alpha(\underline{k}-\underline{k}')\hat{u}_\gamma(\underline{k}')\underline{dk}' ,$$

$$P_{\alpha\gamma} = \delta_{\alpha\gamma} - \frac{k_\alpha k_\gamma}{k^2} \qquad (\delta_{\alpha\gamma} \text{ the Kronecker delta}) ,$$

and the summation convention is in force. The pressure has been eliminated through the use of the equation of continuity

$$(17) \qquad k_\alpha \hat{u}_\alpha = 0 ,$$

giving rise, in the well-known manner, to the projection $P_{\alpha\gamma}$. R is the Reynolds number. We shall now study inertial range solutions of these equations. As before, the viscous dissipation term $(-k^2/R)\hat{\underline{u}}$ is dropped, and the invariance of the equation under the transformation

$$\underline{\hat{u}}' = \underline{\hat{u}}/U$$

(18) $$\underline{k}' = \underline{k}/K$$

$$t' = tUK$$

is noticed; this leads to the conditions

(19) $$\lim_{|\underline{k}|\to 0} |\hat{u}_\alpha(\underline{k})| = \infty \ , \qquad |\hat{u}_\alpha(\underline{k})| \text{ bounded for } |\underline{k}| \neq 0 \ .$$

As before, the integrals $Q_{\alpha\delta}$ must be interpreted in a generalized sense. We rewrite (16) in the symbolic form

(20) $$\partial_t \underline{\hat{u}} = \underline{F}[\underline{\hat{u}}] = \underline{F}[\underline{\hat{u}}](\underline{k})$$

and seek steady solutions of this equation. Consider the case of two-dimensional flow. The equation of continuity (17) is satisfied if

$$\hat{u}_1 = -k_2\rho(\underline{k})$$

$$\hat{u}_2 = k_1\rho(\underline{k}) \ .$$

ρ is a stream function. From (19) we obtain

(21) $$\lim_{k\to 0} |\rho(\underline{k})| = \infty \ , \qquad k = |\underline{k}| \ .$$

A function $\rho(\underline{k})$ invariant under the transformation (18), satisfying (19), and ensuring the boundedness of $\underline{F}[\underline{\hat{u}}]$, is

$$\rho(\underline{k}) = \frac{1}{H(\underline{k})} e^{iL\underline{k}}$$

where $L\underline{k}$ is an arbitrary linear function of $\underline{k}$ and $H(\underline{k})$ is a homogeneous polynomial in k_1 and k_2 which vanishes only when $\underline{k} = 0$. If $H(\underline{k})$ is such that $\underline{F}[\hat{\underline{u}}]$ vanishes at all points of a circle in the (k_1,k_2) plane, then $\underline{F}[\hat{\underline{u}}]$ vanishes everywhere by construction. $\underline{F}[\hat{\underline{u}}]$ can vanish only if $H(\underline{k})$ reflects the fact that equation (16) is invariant under the interchange of the k_1 and k_2 axes; nevertheless $H(\underline{k})$ contains an infinite number of parameters, to be determined by satisfying the infinite number of equations

(22) $$F[\hat{u}](\underline{k}) = 0 , \quad |\underline{k}| = \text{constant} .$$

This will be done as follows: $H(\underline{k})$ will be chosen to be invariant under interchange of k_1 and k_2 and under the transformation

$$k_1' = -k_1$$

$$k_2' = -k_2 .$$

It is then sufficient to satisfy (22) on the portion of the unit circle lying between the lines $k_1 = 0$ and $k_1 - k_2 = 0$. On the unit circle all the functions $H(\underline{k})$ have comparable magnitude. Our effort will thus be directed towards the minimization of appropriate residuals of equation (22).

Two Dimensional Flow

We now proceed to carry out the minimization of $\underline{F}$ in the case of two dimensional flow. We choose n equidistant points $\underline{s}^i = (\cos\theta_i, \sin\theta_i)$, $i = 1,\ldots,n$ on the unit circle, $\theta_i = \theta_o/2 + i\theta_o$,

$0 \leq \theta_i \leq \pi/4$; if $\underline{F}$ can be made to vanish on those n points, it will vanish by symmetry on 8n+4 points of the whole unit circle. We define the total residual relative to $\hat{u}$ to be

$$\text{Total residual} = \sum_i \{|F_1[\underline{\hat{u}}](\underline{s}^i)| + |F_2[\underline{\hat{u}}](\underline{s}^i)|\} .$$

The total residual is minimized first within a one parameter family of functions H, then within families of functions with a larger number of parameters. Each new parameter is introduced in a way which makes use of the information previously obtained.

The point $\theta = 0$ is to be avoided on computational grounds: at this point the singularities of $Q_{\alpha\delta}$ coalesce and an added numerical effort would be required. At the point $\theta = \pi/4$, $\underline{F}$ is always zero by symmetry. Write

$$f_{\delta\gamma,\theta}(\underline{k}') = \hat{u}_\delta(\underline{k}')\hat{u}_\gamma(\underline{s}-\underline{k}') , \qquad \underline{s} = (\cos\theta, \sin\theta) ;$$

the generalized integral

$$Q_{\delta\gamma}(\underline{s}) = \int f_{\delta\gamma,\theta}(\underline{k}')d\underline{k}'$$

is replaced by the equivalent expression

$$\int_D g_{\delta\gamma,\theta}(\underline{k}')d\underline{k}' ,$$

where

$$D = \{k_1', k_2' \mid 0 \leq k_1' \leq \cos\theta, \ 0 \leq k_2' \leq \sin\theta\}$$

and

$$g_{\delta\gamma,\theta} = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} f_{\delta\gamma,\theta}(i \cos \theta + k_1', j \sin \theta + k_2') .$$

$g_{\delta\gamma,\theta}$ is now a bounded function of $\underline{k}'$. The infinite sum is approximated by

$$g_{\delta\gamma,\theta N} = \sum_{i+j\leq N} f_{\delta\gamma,\theta}$$

where N is determined by

$$\sum_{i+j=N} (|f_{11},_\theta| + |f_{12,\theta}| + |f_{22},_\theta|) \leq e \ , \quad e \text{ small.}$$

It is convenient to choose e to be a multiple of $V = \cos \theta \sin \theta$, the area of D.

The integration of g has been carried out by a straightforward second order integration formula, with M points in each direction.

The first, one parameter, family of functions H was chosen to be

$$H(\underline{k}) = k^{\beta_1} , \qquad k = \sqrt{k_1^2 + k_2^2} = |\underline{k}| .$$

The reason for this choice is not only the analogy with the Burgers equation, but also the following argument: for $\partial_t \hat{\underline{u}}$ to vanish, the vector with components $k_\delta Q_{\delta\gamma}$ must be in the null-space of the operator $P_{\alpha\gamma}$, i.e. it must be orthogonal to the unit circle. The closer $\hat{\underline{u}}$ comes to rotational invariance, the more likely it is that this orthogonality will hold. There exist no non-zero rotation-invariant

vectors, but in an intuitive sense this choice of H brings one close to rotational invariance. Additional parameters are introduced as indicated. The spectrum is given by

$$E(k) = \text{constant}\cdot k \int_0^{\pi} \hat{\underline{u}}(\underline{k})\hat{\underline{u}}^*(\underline{k})d\theta \ , \quad \cos\theta = k_1/|\underline{k}| \ , \quad |\underline{k}| = k \ .$$

One may notice that, as in the one-dimensional case, the solutions we shall obtain represent a flow of energy from low to high wave numbers. They are the Fourier transforms of distributions of vortices.

The functions $g_{\delta\gamma,\theta}$ are not well behaved, in particular they are subject to large oscillations around the points $\underline{k}' = 0$ and $\underline{k}' = \underline{s}$. Numerical integration would therefore appear to be a difficult undertaking. Fortunately the functions $g(\underline{k}') + g(\underline{s}-\underline{k}')$ are reasonably smooth, and reliable results can therefore be obtained by an integration formula which is symmetric, i.e. invariant under the transformation $g(\underline{k}') \rightarrow g(\underline{s}-\underline{k}')$ for each $\underline{s}$. e must be so small that the effect of further change in e on the location of the smallest residual is small; more stringent requirements cannot be met in reasonable computing time. Tables II and III demonstrate that the effects of a change in the number of points M in the integration formula and of a change in e are not major. It can also be verified that the effect of a change in the number of points n on the circle on which the residual is computed is minor when $n \geq 3$.

The residuals are small compared to what they are in the case of the Burgers equation. One cannot conclude from this that all

Table II

Minimization of $\underline{F}[\hat{\underline{u}}]$

$H = k^{2\beta_1}$; $e = 0.05V$; $M = 5$; $n = 3$

β_1	Total residue	σ	
1.425	.0180	2.7	
1.400	.0099	2.6	
1.375	.0059	2.5	
1.350	.0048	2.4	←
1.325	.0089	2.3	
1.300	.0147	2.2	
1.275	.0206	2.1	
1.250	.0257	2.0	
1.225	.0324	1.9	
1.200	.0328	1.8	
1.175	.0462	1.7	
1.150	.0557	1.6	
1.125	.0675	1.5	

Table III

Minimization of $\underline{F}[\hat{\underline{u}}]$

$H = k^{2\beta_1}$; $e = 0.02V$; $M = 6$; $n = 3$

β_1	Total residue	σ	
1.550	.0678	3.2	
1.525	.0544	3.1	
1.500	.0435	3.0	
1.475	.0338	2.9	
1.450	.0256	2.8	
1.425	.0187	2.7	
1.400	.0126	2.6	
1.375	.0068	2.5	
1.350	.00332	2.4	←
1.325	.00330	2.3	←
1.300	.0069	2.2	
1.275	.0112	2.1	
1.250	.0156	2.0	
1.225	.0205	1.9	
1.200	.0258	1.8	
1.175	.03191	1.7	

our trial solutions are in some sense nearly solutions of $\underline{F}[\hat{\underline{u}}] = 0$; it should be remembered that if we change the circle on which the minimization is carried out some of the residuals may be considerably amplified. This remark may be used for computing upper bounds and lower bounds for the exponent σ of the inertial range spectrum.

In the numerical tables σ is the inertial range exponent associated with the given trial function H, i.e. the inertial range spectrum would be $Ak^{-\sigma}$ if the trial function H under consideration annihilated $\underline{F}$.

Table IV demonstrates that the addition of parameters to H, i.e. the use of more complicated trial functions H, causes only marginal changes in σ. The conclusion from the tables is that

$$\sigma = 2.3 \pm 0.15 .$$

It is hoped that future work will yield a more accurate determination. The value of σ is almost exactly in the middle between the value $\sigma = 3$ found by Leith [9] and Kraichnan [8] and the value $\sigma = 5/3$ found by Kolmogoroff. Both Kraichnan and Leith state that the values $\sigma = 5/3$ and $\sigma = 3$ can both apply to two-dimensional turbulence, and therefore to a limited extent the present calculation confirms their results.

The coefficient A in the inertial range spectrum depends on the nature of the large scale flow and cannot be determined by the present theory. In particular one can provide examples of large scale flows which do not give rise to an inertial range. One example is the periodic flow

Table IV

Minimization of $\underline{F}[\hat{\underline{u}}]$

$$H(\underline{k}) = (k_1^2 + k_2^2)^{\beta_1}(|k_1|^{2\beta_1} + |k_2|^{2\beta_2}); \; e = 0.01V; \; M = 5; \; n = 4$$

β_1	β_2	total residue	σ	
1.40	0.	.0126	2.6	
1.39	.01	.0180	2.6	
1.38	.02	.0476	2.6	
1.3250	0.	.0076	2.3	
1.3150	.01	.0236	2.3	
1.3125	0.	.0052	2.25	
1.3025	.01	.0258	2.25	
1.3100	-.01	.0307	2.20	
1.3000	0.	.0039	2.20	←
1.2900	.01	.0277	2.20	
1.2850	-.01	.0240	2.10	
1.2750	0.	.0044	2.10	
1.2650	.01	.0323	2.10	
1.2600	-.01	.0172	2.00	
1.2500	0.	.0092	2.00	
2.2400	.01	.0366	2.00	

(23) $\underline{u} = (C \cos \alpha x_1 \sin \alpha x_2, -C \sin \alpha x_1 \cos \alpha x_2)$,

$C, \alpha,$ constant .

$\underline{u}$ is a solution of Euler's equations, $\underline{u}e^{-t/R}$ is a solution of the Navier-Stokes equations for all time t. Energy is not transmitted to the higher modes and therefore A = 0.

Three-Dimensional Flow

It can be seen that a two-dimensional solution of the equation $\underline{F}[\hat{\underline{u}}] = 0$ is also a solution of this equation in three-dimensional space; this solution satisfies the condition

$$\lim_{k \to 0} |\hat{\underline{u}}(\underline{k})| = \infty$$

and for $k \neq 0$ has a finite spectrum E(k); these conditions are natural generalizations of conditions (19), fully consistent with the assumption that k lies in the inertial range.

An attempt was made to find other solutions of $\underline{F}[\hat{\underline{u}}] = 0$ in three dimensions by an appropriate sequence of minimizations. None were found. The reason for the absence of genuinely three-dimensional solutions can be explained as follows: because of the equation of continuity (17), $\hat{\underline{u}}$ must be of the form

$$\hat{\underline{u}} = \underline{k} \times \underline{\rho}$$

where $\times$ denotes a vector product and $\underline{\rho}$ is a vector potential. For any choice of $\underline{\rho}$, $\hat{\underline{u}}$ lies in a plane orthogonal to $\underline{\rho}$, and so does the

vector $\underline{w}$ with components $w_\gamma = k_\delta Q_{\delta\gamma}$. It can be seen that $\underline{w} \neq 0$ for any reasonable choice of $\underline{\rho}$; thus for $F_\alpha[\hat{u}] = ik_\delta P_{\alpha\gamma} Q_{\delta\gamma}$ to vanish $\underline{w}$ must be in the null space of the projection $P_{\alpha\gamma}$, i.e. must be orthogonal to the unit sphere. This is impossible.

In physical terms the fact that the solutions are two-dimensional can be explained as follows: The inertial range flow consists of a random collection of vortices; in the neighborhood of each vortex the fluid flows in a plane orthogonal to the vortex, i.e. it is two-dimensional.

There are clearly differences between the two-dimensional and three-dimensional cases which account for the disorderly appearance of three-dimensional flow. The orientations of the vortices (the normals to the planes of the flows) are random, and it can be argued that the amplitudes of the motions are larger in three dimensions. In particular, no three-dimensional flow similar to (23) is known, such that the amplitude of the corresponding inertial range flow is zero; I conjecture that none exists.

The fact that in the small the solution of the equations of motion is two-dimensional should be less surprising since we already know of a three-dimensional flow which left to itself becomes two-dimensional; this happens in a convection problem, see e.g. [5].

The inertial range spectrum of three-dimensional turbulence is thus of the form

$$E(k) = Ak^{-\sigma} , \quad \sigma = 2.3 \pm 0.15 .$$

This result is at variance with Kolmogoroff's law, which is thus found to be invalid in all cases.

The lack of validity of Kolmogoroff's law is not very surprising. It is not necessary to attach undue importance to results obtained by dimensional analysis, since the possible relations between the various properties of inertial range flow are only dimly understood. It should be remembered that in two space dimensions dimensional arguments yield two distinct spectra.

Inertial range flow should be visualized as follows: vortices are constantly being shed from regions of shear in the main flow. Rapidly they assume an asymptotic form with the spectrum we have just computed. Their amplitude decays under the influence of viscosity, and they are constantly replaced by new vortices. This picture provides an explanation of the independence of the form of the inertial range spectrum from the form of the large-scale motion.

It may be worth pointing out that equation (20) is derived from equation (16) through an argument strongly reminiscent of arguments used in boundary layer theory; in fact equation (20) can be obtained by expanding $\underline{\hat{u}}$ formally in inverse powers of a Reynolds number $\mathcal{R}$ characteristic of inertial range flow. The search for similarity solutions is natural in view of the absence of well-defined length scales in inertial-range flow, and is of course a common device in boundary layer theory.

Comments

The present paper has arisen from an attempt to solve numerically the Green-Taylor problem [14], i.e. from an attempt to study properties of turbulence by examining numerical solutions of initial value problems for the Navier-Stokes equations. Such attempts are doomed to failure. The number of mesh points required is huge (an estimate can be obtained from the fact that the sum defining $g_{\delta\gamma,\theta N}$ contains thousands of terms), and the accumulated errors in the integration in time are unavoidably large (see [6]). In fact, the inertial range spectrum obtained by such means is devoid of physical significance and depends mainly on the difference scheme employed. The study presented in this paper avoids these difficulties. It does however verify a fundamental premise of numerical work on the Green-Taylor problem: it shows that the structure of turbulence can be studied by examining properties of a few typical flows.

The premises of the present work are however at variance with the assumptions of most statistical theories of turbulence. The only statistical argument used is the assertion that if a number of solutions tend to a common limit, so does their average.

Finally, the present paper yields not only an approximation to the form of the inertial range spectrum, but also further information about the nature and structure of inertial range flows; this can be used for evaluating eddy coefficients. The manner in which this is to be done will be the subject of a forthcoming paper.

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Appendix

For the convenience of anyone who would wish either to check or to extend the calculations presented in this paper, the program used to obtain the two dimensional results is reproduced. No particular effort has been expended to make the program very elegant or very efficient.

```
              PROGRAM MTC2(OUTPUT)
            COMMON SIGMA,SIGB,SIGA,TSIGB,BM
9     600 FORMAT(45X,4H TOT, F11.8)
9     999 FORMAT(7H SPHERE)
9     009 FORMAT(/1X,4F11.7)
9     071 FORMAT(4H X,Y, 2F11,7)
9     074 FORMAT(10X,2HBM,F11.7)
8     700 FORMAT(20X, 3F11.7)
9     011 FORMAT ( 6H POWER,F11.7)
8     000 FORMAT(1X,2I3, 3F11.7)
9     099 FORMAT(1X,I5,3F11.7)
9     002 FORMAT (2H X,4F11.7)
8     800 FORMAT(1X,4I5)
9     070 FORMAT (1X, 5H*****)
              DSIGB=0.04
              DSIGB=DSIGB/4.
              NSA=3
              NMB=1
             AA=-5.
                 DSIGMA=1.
             NE=4
             SIGMA=0.75-AA/4.
               DSIGMA=DSIGMA/4.
               IFIC=10000
              NBB=1
        EPS=0.05
              PI=3.1415926536
             NM=6
              NT=NM*NM

             DALPHA=PI/14.
               ALPHA=DALPHA/2.

            CHANGE EXPONENT
               DO 7 IE=1,NE
               SIGMA=SIGMA-DSIGMA
               SIGB=-DSIGB
             DO 7 IBB=1,NBB
             SIGB=SIGB+DSIGB
                 TSIGB=SIGB*2.
               SIGA=SIGMA-SIGB
        POW= 3.-4.*SIGMA
        PRINT  9011,POW
              PRINT 8700, SIGMA,SIGA,SIGB
              PRINT 8007, NT
        CNT=NT
        CCNT=CNT*SQRT(CNT)
              CQNT=SQRT(CNT)
               DBM=0.1
```

```
      BM=-DBM
      DO 7 JMD=1,NMD
      BM=BM+DBM
      PRINT 9074,BM
      GA=0.
      GB=0.
      GAS=0.
      GBS=0.
C     SPHERICAL AVERAGE
      TOTS=0.
      TOT=0.
      DO 6 JNSA=1,NSA
      ALPHA=ALPHA+DALPHA
      X=COS(ALPHA)
      Y=SIN(ALPHA)
      DX=X
      DY=Y
      DVOL=DX*DY
      PRINT 9071, X,Y
C     CONSTRUCT PROJECTION OPERATOR
      RR=X*X+Y*Y
      PAA=(1.-X*X/RR)
      PAB=-X*Y/RR
      PBB=(1.-Y*Y/RR)
      PAAA=X*PAA
      PAAB=X*PAB
      PABB=X*PBB
      PBAA=Y*PAA
      PBAB=Y*PAB
      PBBB=Y*PBB
C     END PROJECTION
C
C     PREPARE QUANTITIES
      COMAT=0.
      COMBT=0.
      VCA=0.
      VCB=0.
C
C     NEW TRIAL
      NPA=0.
      NPB=0.
      CMN=NM
      HX=DX/CMN
      HY=DY/CMN
      HX0=0.5*HX
      HY0=0.5*HY
      DO 1 II=1,NM
      DO 1 JJ=1,NM
      CCI=II-1
```

```
    CCJ=JJ-1
     CSA=HXO+CCI*HX
       CSB=HYO+CCJ*HY
COMA=0.
COMB=0.


NEW SUM

         QBB=0.
         QAB=0.
       QAA=0.
      DO 10 N=1,IFIC
RAA=0.
RAB=0.
RBB=0.
       TEST=0.
DO 99 I=1,N
      J=N-I+1
        CI=I
       CIM=I-1
        CJ=J
       CJM=J-1
C     I POS, J POS
       XK=CSA+CIM*DX
          YK=CSB+CJM*DY
       XKK=X-XK
          YKK=Y-YK
           CALL U(XK,YK,UA,UB)
          CALL U(XKK,YKK,UAP,UBP)
RAA=RAA-UA*UAP
RAB=RAB-UA*UBP
RBB=RBB-UB*UBP
C      I NEG, J NEG
         XK=CSA-CI*DX
         YK=CSB-CJ*DY
       XKK=X-XK
          YKK=Y-YK
           CALL U(XK,YK,UA,UB)
          CALL U(XKK,YKK,UAP,UBP)
RAA=RAA-UA*UAP
RAB=RAB-UA*UBP
RBB=RBB-UB*UBP
C      I NEG, J POS
         XK=CSA-CI*DX
          YK=CSB+CJM*DY
       XKK=X-XK
          YKK=Y-YK
           CALL U(XK,YK,UA,UB)
```

```
               CALL U(XKK,YKK,UAP,UBP)
      RAA=RAA-UA*UAP
      RAB=RAB-UA*UBP
      RBB=RBB-UB*UBP
C              I POS, JNEG
             XK=CSA+CIM*DX
               YK=CSB-CJ*DY
             XKK=X-XK
                YKK=Y-YK
                 CALL U(XK,YK,UA,UB)
                CALL U(XKK,YKK,UAP,UBP)
      RAA=RAA-UA*UAP
      RAB=RAB-UA*UBP
      RBB=RBB-UB*UBP
   99 CONTINUE
                 QAB=QAB+RAB
                 QAA=QAA+RAA
                QBB=QBB+RBB
          TEST=ABS(RAA)+ABS(RAB)+ABS(RBB)
                  IF(TEST.LE.EPS)   GO TO 11
   10 CONTINUE
   11 CONTINUE
C     END SUM
C
C
C              PERFORM PROJECTION
C      X- COMPONENT
      COMA   = COMA
     1 -PAAA*QAA-PAAB*QAB
     2 -PBAA*QAB-PBAB*QBB
C      END X-COMPONENT
C
C      Y- COMPONENT
      COMB   = COMB
     1 -PAAB*QAA-PABB*QAB
     2 -PBAB*QAB-PBBB*QBB
C        END Y-COMPONENT
          PRINT 8000,II,JJ,COMA,COMB
            IF(COMA)  701,701,700
  700 NPA=NPA+1
  701 CONTINUE
           IF(COMB)  703,703,704
  704 NPB=NPB+1
  703 CONTINUE
C
      COMAT=COMAT+COMA
      COMBT=COMBT+COMB
      VCA=VCA+COMA*COMA
      VCB=VCB+COMB*COMB
```

```
1 CONTINUF
  END TRIALS
       NGA=NT-NPA
        NGB=NT-NPR
      PRINT 8800, NPA,NGA,NPB,NGB

       COMAT=DVOL*COMAT
        COMBT=DVOL*COMBT
      VCA=DVOL*DVOL*VCA
         VCB=DVOL*DVOL*VCB

  COMAT=COMAT/CNT
  COMRT=COMBT/CNT
  VCA=VCA/CCNT
  VCB=VCB/CCNT
         VCA=VCA-COMAT*COMAT/CCNT
             VCR=VCR-COMBT*COMBT/CCNT
      PRINT 9009, COMAT,COMBT,VCA,VCB
      GA=GA+ABS(COMAT)
         GB=GB+ABS(COMBT)
       GAS=GAS+ABS(VCA)
        GBS=GBS+ABS(VCB)
6 CONTINUE
        PRINT 9999
       PRINT 9009, GA,GB,GAS,GBS
     TOT=GA+GB
        PRINT 9600, TOT
        TOTS=GAS+GBS
         PRINT 9600, TOTS
  PRINT 9070
7 CONTINUF
       CALL EXIT
  END


          SUBROUTINE U(X,Y,RA,RB)
     COMMON SIGMA,SIGB,SIGA,TSIGB,BM
       RR=X*X+Y*Y
        STR=RR**SIGA
     XA=ABS(X)
        YA=ABS(Y)
       STRB=XA**TSIGB+YA**TSIGB
         STRB=STRB+BM*((XA*YA)**SIGB)
        STR=STR*STRB
         STR=1./STR
        RA=-Y*STR
        RB=X*STR
```

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